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LETTER TO THE EDITOR

Fluctuation-induced second-order phase transitions

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Abstract. In the framework of the field-theoretical renormalisation group, we study under which conditions a second-order transition can be present in systems which should only undergo first-order phase transitions, according to the Landau criterion.

In the framework of the Landau theory (Landau and Lifshitz 1958), phase transitions are characterised by the order parameter φ_i $(i = 1, \ldots, N)$; one can construct an effective free energy $F(\varphi)$, which is a polynomial in φ, and the value of φ is established by minimising $F(\varphi)$.

In the high-temperature phase $\varphi = 0$ (no linear term in φ is present in F); for small values of φ, $F(\varphi)$ can be expanded as

$$F(\varphi) \sim \tau\varphi^2 + u_3\varphi^3 + u_4\varphi^4 + \mathrm{O}(\varphi^5). \tag{1}$$

If $u_3 = 0$ we have a second-order phase transition at $\tau = 0$ $(\xi = 1/\tau^{1/2} \to \infty)$; if $u_3 \neq 0$ a first-order phase transition is present at $\tau > 0$.

It is generally assumed that if there are no symmetries, which impose $u_3 = 0$ (as happens in the ferromagnetic case), u_3 is different from zero (unless at some special points). Therefore by studying the symmetries of the problem and trying to construct a cubic invariant, we can distinguish systems having a second-order transition (no cubic invariants) from those having a first-order transition (a cubic invariant is allowed). This classification is very useful: however there are notable exceptions: some ferromagnets undergo a first-order transition, and in the Potts model with q states $(q > 2)$ second-order transitions are allowed (Baxter 1973, Nakanishi and Stanley 1981 and reference therein). The first phenomenon (fluctuation-induced first-order phase transitions) is well understood (Bak *et al* 1976, 1977, Mukamel and Krinsky 1976).

Our aim is to study the second phenomenon (fluctuation-induced second-order phase transitions) in the framework of the field-theoretical renormalisation group. In the rest of the paper we will mainly study the Potts model, although many arguments have a more general basis.

It is known that the q-state Potts model ($q = 2$ is the Ising model) has a second-order phase transition for $q \leqslant 4$ and a first-order phase transition for $q > 4$ in two dimensions, on a two-dimensional square lattice, with only nearest-neighbour interaction.

Let us call q_c the maximum value of q for which there is a phase transition for a fixed lattice ($q_c = 4$ for the square lattice, $2 < q_c < 3$ for the cubic lattice). Our aim would be to compute the value of q_c using the field-theoretical version of the renormalisation group

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(Amit 1976, de Alcantara Bonfim *et al* 1980, Priest and Lubensky 1976, Wallace and Zia 1975).

However in this framework it is possible to compute only universal quantities. Some reflection will tell us that q_c cannot be universal: let us consider a square lattice Potts model where the interaction has radius R, i.e. all the spins at distance less than R interact pairwise ($R = 1$: nearest-neighbour interaction; $R = \infty$: infinite-range interaction).

For $q > 2$ when $R \to \infty$ the mean field theory is exact and the transition is first order; for finite R there are corrections proportional to inverse powers of R, and these corrections remain finite at the first-order transition. The situation for the $1/R$ expansion is qualitatively different from a second-order phase transition where the coherence length ξ goes to infinity and the two limits $\xi \to \infty$ and $R \to \infty$ do not commute.

Therefore $q_c(R) \xrightarrow[R\to\infty]{} 2$ and q_c cannot be universal as long as $q_c(1) = 4$. However, if we change the form of the lattices and of the interactions there will be a maximum value of q_c (q_M) for which we have always $q_c \leqslant q_M$. Obviously q_M is universal by definition and it has the meaning of the maximum value of q for which a second-order phase transition is possible.

Let us present our strategy for computing q_M. The critical behaviour of the Potts model can be studied in $6-\varepsilon$ dimensions. In this case for $q < \frac{10}{3}$ one finds a non-trivial fixed point of φ^3 type. The effective free action $F(\tau, \varphi)$ (where $\tau = T = T_c$) near the critical point scales like

$$F(\tau, \varphi) \sim \tau^{D\nu} f(\varphi \tau^{\nu(2-D)-\eta}) \tag{2}$$

with $f(0) = 0$ and $\tau^\nu = \xi^{-1}$.

The function f can be computed in $6-\varepsilon$ dimensions and turns out to behave like

$$f(z) = z^2 + \varepsilon^{1/2} z^3 + \mathrm{O}(\varepsilon^2 z^4). \tag{3}$$

Stability implies that $z = 0$ is the absolute minimum of the function $f(z)$; if not, $z = 0$ is a metastable phase and a first-order transition should develop for positive τ.

Our consistency criterion to see if a second-order phase transition is possible is to compute the potential f in the scaling region and to look for the absolute minimum (if any).

In the Potts model, in the ε expansion, the fixed point is of φ^3 type and one can argue that for this kind of fixed point $f(-\infty) = -\infty$ in all dimensions. At this stage we should conclude that all Potts models ($q > 2$) have a first-order transition. However this conclusion would be premature. The φ^3 fixed point may become unstable (we distinguish between an unstable fixed point whose domain of abstraction has zero measure, and thermodynamic instability, i.e. existence of lower free energy states), decreasing the dimension, and different fixed points may become relevant. If that happens we should compute the function f on the new fixed point.

There are two possibilities:

(a) there is a value of z such that $f(z) < 0$ ($f(0) = 0$);

(b) $f(z) \geqslant 0 \qquad \forall z$.

If (a) is valid the new fixed point is thermodynamically unstable and the transition will be always first order; if (b) holds a second-order phase transition is possible. As we have stressed before, a second-order phase transition will be realised, in an actual system, only if the system does not undergo a first-order transition before reaching the fixed point.

The shape of the function $f(z)$ will tell us the value of q_{M}. In order to implement our ideas in the Potts model we have considered a three-coupling-constant theory defined by

$$H=\int \mathrm{d}^D x\ (\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0\varphi^2+(1/3!)Q_{ijk}\varphi_i\varphi_j\varphi_k\lambda_1$$
$$+(1/4!)\lambda_2 F_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l+(1/4!)\lambda_3 S_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l) \tag{4}$$

where $i, j, k, l=1,\ldots,N$. The definitions of the tensors are

$$Q_{ijk}=\sum_\alpha l_i^\alpha l_j^\alpha l_k^\alpha, \qquad F_{ijkl}=\sum_\alpha l_i^\alpha l_j^\alpha l_k^\alpha l_l^\alpha, \qquad S_{ijkl}=\tfrac{1}{3}(\delta_{ij}\delta_{kl}+\text{permutations}),$$

where $\alpha_i=1,\ldots,N-1$, and the l_i^α are such that $\Sigma_\alpha l_i^\alpha=0$; $\Sigma_\alpha l_i^\alpha l_j^\alpha=(N+1)\delta_{ij}$, $\Sigma_i l_i^\alpha l_i^\beta=(N+1)\delta^{\alpha\beta}-1$. Our aim would be to find the corresponding fixed point.

It is clear that this problem is very difficult to solve in the standard ε expansion. To study the φ^3 interaction we must stay in $6-\varepsilon$ dimensions and to study the φ^4 interaction we must stay in $4-\varepsilon$ dimensions.

We use therefore the fixed-dimension formalism (Parisi 1980): one defines a renormalised field φ_{R} proportional to the field φ such that

$$\langle\varphi_{\mathrm{R}}\varphi_{\mathrm{R}}\rangle=[p^2+m^2+\mathrm{O}(p^4)]^{-1} \tag{5}$$

in momentum space, where $m^{-1}=\xi$; the renormalised dimensionless coupling constants are

$$g_1=\Gamma_{\mathrm{R}}^{(3)}/m^{6-D},$$
$$g_2=\Gamma_{\mathrm{R}(F)}^{(4)}/m^{4-D}-\Gamma_{\mathrm{R}(F)}^{(4)}/m^{4-D}|_{\lambda_2=\lambda_3=0}, \qquad g_3=\Gamma_{\mathrm{R}(S)}^{(4)}/m^{4-D}-\Gamma_{\mathrm{R}(S)}^{(4)}/m^{4-D}|_{\lambda_2=\lambda_3=0}. \tag{6}$$

The subtraction in the definition of the couplings g_2, g_3 is such that $g_2=g_3=0$ at $\lambda_2=\lambda_3=0$.

With these preliminaries we can define the function β_i as

$$\beta_i=m^2\,\mathrm{d}g_i/\mathrm{d}m^2+\tfrac{1}{2}N_i\gamma(g_1)g_i \tag{7}$$

where $i=1,2,3$; N_i is such that $N_1=3$, $N_2=N_3=4$ and $\gamma(g)$ is the usual renormalised function which defines the critical index η for the Potts model. When $m^2\to 0$ the renormalised coupling constants tend toward a stable simultaneous zero of the functions. The stability condition is that the Hessian matrix for $\tilde{\beta}_i=-\beta_i$

$$H_{ik}=\partial\tilde{\beta}_i/\partial g_k \tag{8}$$

must have all negative eigenvalues at the fixed point.

Up to this point everything is exact. The main problem consists in computing the functions β_i. It is known that the β_i functions can be expanded in powers of the renormalised coupling constant, with computable coefficients, in terms of Feynman-type diagrams (figure 1). For the pure φ^4 theory in three dimensions the one- or two-loop approximations give results up to the correct order of magnitude for the fixed point and the critical exponents (Parisi 1980), while a long six-loops computation gives very precise results (Baker *et al* 1978). The corresponding two- and three-loop computations have already been done for the pure φ^3 theory (Fucito and Marinari 1981a, b).

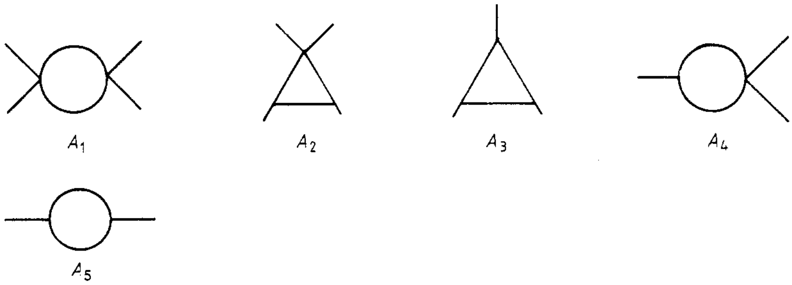

Figure 1. Diagrams involved in our one-loop computation.

In this note we present the one-loop result: although we cannot hope to be very quantitative, in such a crude approximation, we are confident that the qualitative picture should be correct. The one-loop $\tilde{\beta}_i$ functions are:

$$
\begin{aligned}
\tilde{\beta}_1 = -\beta_1 = {} & \frac{6-D}{4} g_1 - \tfrac{3}{4}(N+1)(N-1)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right) g_1 g_3 - \frac{1}{2}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right) g_1 g_2 \\
& + \frac{1}{4}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right)(N+1)^2(N-2) g_1^3 \\
& - \tfrac{1}{16}(N+1)^2(N-2)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right) g_1^3, \\
\tilde{\beta}_2 = -\beta_2 = {} & \frac{4-D}{2} g_2 - \frac{1}{12}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right)(N+8) g_2^2 - \frac{N(N+1)}{2}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right) g_2 g_3 \\
& - \tfrac{3}{4}(N+1)^2\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right) g_3^2 + \tfrac{3}{2}(N+1)^3\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right) g_1^2 g_3 \\
& + \tfrac{1}{2}(N+1)^2(N-3)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right) g_1^2 g_2 \\
& - \tfrac{1}{12}(N+1)^2(N-1)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right) g_1^2 g_2, \\
\tilde{\beta}_3 = -\beta_3 = {} & \left(\frac{4-D}{2}\right) g_3 - \tfrac{3}{4}(N+1)(N-1)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right) g_3^2 \\
& + \tfrac{3}{2}(N+1)^2(N-2)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right) g_1^2 g_3 + (N+1)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right) g_1^2 g_2 \\
& - \tfrac{1}{12}(N+1)^2(N-1)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right) g_1^2 g_3 - \Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right) g_2 g_3.
\end{aligned}
\tag{9}
$$

In table 1 are specified the multiplicities and the numerical values, in table 2 the tensorial couplings, of the diagrams of figure 1 which are involved in this computation. In figure 1 the diagrams are labelled with A_i, $i = 1, \ldots 5$, and in table 2 the tensorial couplings are defined by A_1^{SF}, A_1^{SS}, A_2^{QQF} etc, to specify that the vertices of A_1 can be coupled with one tensor F and one tensor S, with two tensors F_{ijkl}, or with two tensors S_{ijkl} and analogously for the other diagrams.

In table 3 there is a detailed list of some values of g_1, g_2, g_3 for which equations (9) have a simultaneous stable zero. The values of q and D for which there is a slash in the

Table 1. Multiplicities (in brackets) and values of the diagrams of figure 1 as they enter the definitions of formula (7).

$$A_1: \left(\frac{3}{2}\right)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right)\frac{1}{2}, \qquad A_2: (6)\frac{1}{4}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right), \qquad A_3: (1)\frac{1}{4}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right),$$

$$A_4: \left(\frac{3}{2}\right)\frac{1}{2}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{6-D}{2}\right), \qquad A_5: \left(\frac{1}{2}\right)\frac{1}{12}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right).$$

Table 2. Tensorial couplings for the graphs of figure 1. For simplicity we have omitted the indices of the tensor.

$$A_1^{SS}: \frac{(N+8)}{g}S, \qquad A_1^{FF}: (N+1)^2S+(N+1)(N-1)F, \qquad A_1^{FS}+A_1^{SF}: \tfrac{2}{3}N(N+1)S+\tfrac{4}{3}F,$$

$$A_2^{QQF}: (N+1)^3S+(N+1)^2(N-2)F, \qquad A_2^{QQS}: \tfrac{1}{3}(N+1)^2(N-3)S+\tfrac{2}{3}(N+1)F,$$

$$A_3^{QQQ}: (N+1)^2(N-2)Q, \qquad A_4^{QS}: \tfrac{2}{3}Q, \qquad A_4^{QF}: (N+1)(N-1)Q,$$

$$A_5^{QQ}: (N+1)^2(N-1).$$

Table 3. Values of the coupling constant for various q and D, for which the functions β_i have a simultaneous stable fixed point. The values of q and D for which there is a slash are those for which no stable fixed points are allowed (fixed points are allowed but they are unstable).

D \ q		4.0	3.0	2.0	1.0
	$g_1 =$	0.1546	0.3024	0.6001	0.9073
4.5	$g_2 =$	0.091	0.1742	0	0
	$g_3 =$	0.434	0.0646	0	0
	$g_1 =$	0.1742	0.3012	0.7071	1.069
4.0	$g_2 =$	0.1361	0.3197	0.0149	0
	$g_3 =$	0.0792	0.1285	0.4961×10^{-2}	0
	$g_1 =$	1.322	0.2877	0.7105	1.171
3.5	$g_2 =$	0.1280	0.4625	0.2385	0
	$g_3 =$	0.1217	0.2048	0.0808	0
	$g_1 =$		0.2632	0.6849	1.206
3.0	$g_2 =$	/	0.5428	0.5027	0
	$g_3 =$		0.2778	0.1731	0
	$g_1 =$		0.2300	0.6311	1.171
2.5	$g_2 =$	/	0.5000	0.7065	0
	$g_3 =$		0.3326	0.2469	0
	$g_1 =$		0.1905	0.5523	1.069
2.0	$g_2 =$	/	0.1970	0.7800	0
	$g_3 =$		0.3825	0.2766	0

table are those for which no stable fixed point is allowed. In figure 2 we can see the region where the Potts model has non-trivial stable fixed points ($g_1 \neq g_2 \neq g_3 \neq 0$) and the region where only the pure φ^3 interaction points are stable. We now notice that the solutions of equations (9) such that g_2 and g_3 are the same as those of table 3, but g_1 has opposite sign, also represent an equivalent stable solution.

Summarising, one finds that there is a region (shown in figure 1) where the φ^3–φ^4 fixed point is stable and the pure φ^3 point is unstable.

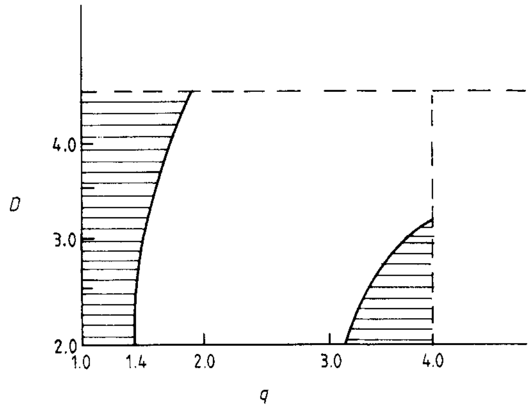

Figure 2. We show our results for $2 \leqslant D \leqslant 4.5$.

What happens to the potential $f(\varphi)$ at the fixed point? The precise computation for $f(\varphi)$ which incorporates the correct asymptotic behaviours at large φ is quite problematic.

For simplicity we have decided to use a zero-loop approximation, i.e. to write $f(\varphi)$ as

$$f(\varphi) = \tfrac{1}{2}m^2\delta_{ij}\varphi_i\varphi_j + (1/3!)g_1Q_{ijk}\varphi_i\varphi_j\varphi_k + (1/4!)(g_2F_{ijkl} + g_3S_{ijkl})\varphi_i\varphi_j\varphi_k\varphi_l \tag{10}$$

neglecting the loop corrections.

As far as we know, the breaking will happen in the direction $\varphi_i = \varepsilon_i^1|\varphi|$ where ε_i^1 is the ith vector which points in the direction 1 (arbitrary) of a hypertetrahedron in an N-dimensional space and whose normalisation rules have already been explained. We can study under which conditions the function $f(\varphi)$ is positive definite as a function of $|\varphi|$.

One finally finds (solving (10)) the condition

$$(N^2-1)^2g_1^2 - 3[g_3N + g_2(N^3+1)] < 0, \tag{11}$$

which turns out to be satisfied in the whole region where the $\varphi^4 - \varphi^3$ fixed point exists and is stable.

According to our findings there exists a large area in the D–q plane where the Potts model may have a second-order phase transition.

This area is much larger than the region where the model with nearest-neighbour interaction has actually a second-order transition, especially in dimensions 3 and 4.

These results suggest that different realisations on the lattice (e.g. ferromagnetic first-neighbour coupling and small antiferromagnetic second-neighbour coupling) may show a rather different pattern of transitions. The approach described in this note opens the possibility of computing, using field-theoretical methods, the critical exponents of the Potts model when it undergoes a second-order phase transition.

In order to have semi-quantitative prediction for the exponents we should need a two-loop calculation.

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